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Wave propagation and boundary instability in erodible-bed channels

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Wave propagation in one-dimensional erodible-bed channels is discussed by using the shallow-water approximation for the fluid and a continuity equation for the bed. In addition to gravity waves, a third wave, which gives the velocity of propagation of a bed disturbance, is found. An appropriate dimensional analysis yields the quasi-steady approximation for the complete shallow-water equations.

The well-known linear stability analysis of free-surface flows is extended to include the erodibility of the bed. The critical Froude number $\mathbf{F}_c$ above which the free-surface of the fluid may become unstable is obtained. It is shown that erodibility increases the stability of the free surface, in qualitative agreement with previous experiments if $q_b > q_s$, q_b and q_s being respectively the contact-bed discharge and suspended-material discharge. The stability theory is also used to discuss coupled beds and surface waves. From it, five different configurations have been obtained: a sinusoidal wave pattern moving downstream, a transition zone and antidunes moving upstream, moving downstream and stationary. These bed forms are in agreement with experimental results; hence shallow-water theory seems to give a reasonable explanation of the boundary instability.

It is shown that the quasi-steady approximation and Kennedy's (1963) stability analysis will be in agreement if $(kh)^2 \ll 1$, where k is the wave number, and h is the depth of the water. When the phase shift δ is introduced in the quasi-steady approximation, the five bed patterns derived from the full equations are found again.

1. Introduction

It is well known (e.g. Vanoni & Brooks 1957; Simons & Richardson 1961), that the boundary surface S_L that separates an erodible stream bed from the water flowing above may become unstable. As a consequence of this instability, different bed forms, sometimes coupled with surface waves, appear in natural streams and experimental flumes. The different bed-form domains can be classified according to the flow régime into three categories. (α) Lower régime: ripples, 'incipient' flat bed, ripples on dunes; all these waves move downstream. (β) Transition régime: wash-out dunes and flat bed. (γ) Upper régime: flat bed;

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antidunes moving upstream and downstream, stationary waves, chutes and pools.

Note that the incipient flat-bed zone has been reported by Chabert & Chauvin (1963) when the particles of the bed satisfy the inequality

$$d_* = d_m(g\Delta\rho/\rho\nu^2)^{\frac{1}{3}} > 15$$

(or $d_m > 0{\cdot}6$ mm for sands) where d_m is the mean diameter of the bed material, ρ and ν are the density and kinematic viscosity of the water, $\Delta\rho = \rho_b - \rho$, ρ_b is the bed density. This zone was also observed by Liu (1957) and Maggiolo & Borghi (1965). The existence of this zone has been questioned by other authorities, e.g. Knoroz (1959).

The mathematical analysis of the different waves that can propagate in erodible-bed channels has been previously discussed by means of two different approaches: the hydraulic and potential flow models. In the former, due to Exner (1925), the motion of the fluid is described by the steady one-dimensional open-channel equation and the bed behaviour is represented by an empirical formula for the bed-load discharge and an equation of conservation of mass for the balance of the convected material. In the latter, mainly due to Kennedy (1963), who successfully improved Anderson's (1953) model, the fluid is treated as irrotational and inviscid and the influence of the bed is discussed using Exner's approach. These two models have been also discussed by Reynolds (1965), who extended the potential model to two-dimensional wave problems.

It will be shown in the present work how it is possible to give a reasonable explanation of bed-wave propagation and boundary instability in one-dimensional channel flows by means of the fully unsteady shallow-water approximation for the fluid and Exner's equation for the bed. The three different models will be related in a way which makes possible the understanding of their different ranges of application.

2. Shallow-water flows over erodible beds

2.1. *Basic equations*

Let us consider a free-surface flow moving over an erodible bed as shown in figure 1. It is supposed that the order of magnitude H of the depth h is much smaller than the order of magnitude L of the characteristic length in the horizontal direction. Hence, the flow may be considered as shallow.

According to Friedrichs' (1948) lowest-order approximation, the one-dimensional shallow-water equations of motion and conservation of mass read as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial}{\partial x}(h+e) + c_b\frac{u|u|}{h} = 0, \qquad (1)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \qquad (2)$$

where u is the mean velocity of the flow, h is the water depth and e is the local bed elevation measured from a fixed plane of reference. Note that a resistance term $c_b u|u|/h$ is added.

The dimensionless resistance coefficient c_b is a function of the roughness of the bed. It will be assumed that

$$c_b = \Lambda(d_m/h)^n, \tag{3}$$

where n is a positive number and Λ is an empirical constant. When $n = \frac{1}{3}$ and $\Lambda = g/441$ when g is measured in m/sec^2, (3) is the well known Strickler's empirical formula, which is assumed valid when the bed configuration is flat; otherwise, the bed roughness will depend on the particular configuration of the bed (e.g. Knoroz 1959). Alternatively, resistance formulae based on the logarithmic distribution of velocity may also be used in the analysis.

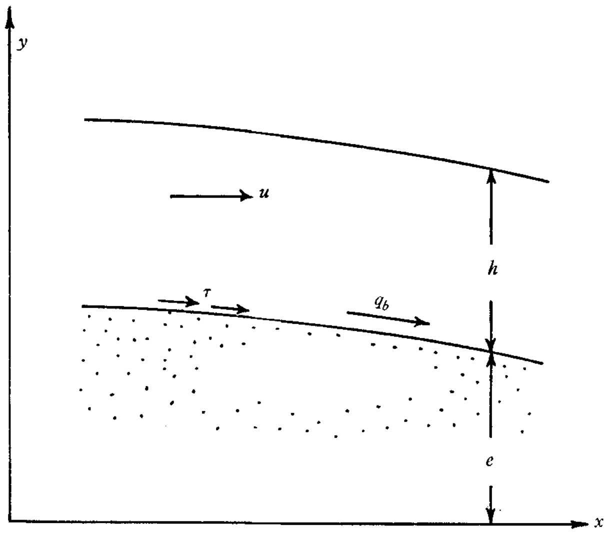

FIGURE 1. Symbols in shallow water theory.

To complete the formulation of the theory, it is necessary to establish the erodible-bed equations. As no theory has yet been able to give a satisfactory explanation of the movement of sediment by flowing water, it will be assumed that the movement of bed particles that slide and roll along the bed can be represented by the bed-load (contact-bed discharge) q_b, in general a function of u and h. Several empirical formulae for q_b are available; the formula

$$\left.\begin{aligned} q_b &= \chi(|\tau| - \tau_0)^r \operatorname{sign} \tau \quad (|\tau| \geqslant \tau_0), \\ q_b &= 0 \quad (|\tau| \leqslant \tau_0) \end{aligned}\right\} \tag{4}$$

will be used in this work, as an example (cf. U.S. Waterways Exp. St. 1935). Here $\tau = \rho c_b u|u|$ is the shear stress transmitted by the fluid to the bed, χ and r are empirical constants, $\tau_0 = g\Delta\rho d_m \tau_0^*$ is the threshold (yield) shear stress of the bed material, and τ_0^* is obtained from Shields's (1936) diagram. It will be assumed in this work that $|q_b| > 0$; this implies that $|\tau| > \tau_0$ in the formula (4). If the fluid also carries material in suspension, the suspended-bed-material discharge q_s must be added to the bed-load discharge q_b.

When $\chi = 8/(g\rho^{\frac{1}{2}}\Delta\rho p)$ (p = porosity of bed material), and $r = \frac{3}{2}$, (4) is the well known Meyer-Peter & Muller (1948) formula expressed in terms of volume of transported material (q_b is given in m^2/sec if τ is given in ton/m^2).

The equation of conservation of mass of the bed is given by

$$\frac{\partial q_b}{\partial x}+\frac{\partial e}{\partial t} = 0.$$

Since $q_b = q_b(u, h)$, this yields

$$q_1(u,h)\frac{\partial u}{\partial x}+q_2(u,h)\frac{\partial h}{\partial x}+\frac{\partial e}{\partial t} = 0, \tag{5}$$

where

$$q_1 = \frac{\partial}{\partial u}q_b, \quad q_2 = \frac{\partial}{\partial h}q_b.$$

2.2. *Dimensionless equations*

Equations (1), (2) and (5) can be written in dimensionless form. Let $u' = u/U$, $h' = h/H, e' = e/H, x' = x/L, t' = t/T, q_b' = q_b/Q_b$, where U, H, L, T, Q_b are characteristic dimensional constants so chosen that u', h', e' and their first derivatives with respect to x', t' are of order unity. For example, L could be a characteristic wavelength, T the period of an oscillation, U and H the mean velocity and depth at a characteristic section of the channel. The characteristic bed discharge Q_b is so chosen that

$$Q_b = \chi(C_b\rho U^2-\tau_0)^r,$$

where C_b is the roughness at the same section, i.e. $c_b = (H/h)^n C_b$. An alternative expression for Q_b can be obtained if we replace $C_b\rho U^2$ by $g\rho HS_0$, where S_0 is the slope of the characteristic flow.

Substitution of the new variables into equations (1), (2), (5) yields three dimensionless equations which may be written in matrix form

$$\mathbf{A}\frac{\partial \mathbf{v}'}{\partial x'}+\mathbf{B}\frac{\partial \mathbf{v}'}{\partial t'}+\mathbf{c}' = 0, \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{F}^2u' & 1 & 1 \\ h' & u' & 0 \\ q_1'\mathbf{M} & q_2'\mathbf{M} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{F}^2\mathbf{T} & 0 & 0 \\ 0 & \mathbf{T} & 0 \\ 0 & 0 & \mathbf{T} \end{bmatrix},$$

$$\mathbf{v}' = [u', h', e'],$$

$$\mathbf{c} = \left[\mathbf{J}\frac{u'|u'|}{h'^{(1+n)}}, 0, 0\right],$$

$$q_1' = \frac{1}{Q_b}\frac{\partial q_b}{\partial u'},$$

$$q_2' = \frac{1}{Q_b}\frac{\partial q_b}{\partial h'},$$

and

$$\mathbf{F}^2 = \frac{U^2}{gH}, \quad \mathbf{T} = \frac{L}{UT}, \quad \mathbf{J} = \frac{C_bLU^2}{gH^2}, \quad \mathbf{M} = \frac{Q_b}{UH}. \tag{7}$$

The dimensionless numbers **F**, **T**, **J** are respectively the Froude, Strouhal and friction numbers, and they also appear in the theory of shallow-water flows over a rough but rigid bed. On the other hand, the number **M** seems to be new in the literature and will be called erosion number, since it gives an estimate of the erosive capacity of the flow. This number can be related to the concentration of total (bed + suspended) sediment discharge C_t, which is defined as the ratio of the total weight of solids to the weight of the water-sediment mixture and is expressed in ppm by the formula

$$\mathbf{M} = \frac{\rho}{p\rho_b}\frac{C_t}{1-C_t}.$$

The highest value of C_t recorded in the extensive experimental work done at Colorado State University (Guy, Simons & Richardson 1966) is $C_t = 49{,}300$ ppm (table 7, run 31, chute-pool configuration), which corresponds to $\mathbf{M} \approx 0{\cdot}03$. Hence, it can be assumed that $\mathbf{M} \ll 1$ in most cases. However, concentrations of over 600,000 ppm have been observed in streams carrying an appreciable amount of sediments at supercritical flows characterized by violently breaking antidunes (chutes and pools), as reported by Nordin (1963).

It will be convenient to define also a bed shear number

$$\mathbf{H} = \frac{\tau_0}{C_b \rho U^2},$$

and a modified erosion number $\mathbf{M}^* = \mathbf{M}/(1-\mathbf{H})$. These appear when (4) is used. It is assumed thoroughout that $\mathbf{H} < 1$.

2.3. *Wave propagation*

It may be asked what happens when a gravity wave propagates over a bed constituted by a granular erodible material. To answer this question, let us determine whether the system (6) has real characteristics, i.e. curves in the (x', t')-plane along which the first partial derivatives of the dependent variables are not uniquely determined. These characteristic curves are the solutions of the ordinary differential equations:

$$\left(\frac{dx'}{dt'}\right)_i = \frac{T}{L}\left(\frac{dx}{dt}\right)_i = s_i \quad (i = 1, 2, 3); \tag{8}$$

s_i are the eigenvalues of the problem $\mathbf{A} - s\mathbf{B} = 0$, whose secular equation reads

$$\mathbf{F}^2(\mathbf{T}^3 s^3 - 2u'\mathbf{T}^2 s^2) - \mathbf{T}s\,(q_1'\mathbf{M} + h' - u'^2\mathbf{F}^2) + (q_1'u' - q_2'h')\,\mathbf{M} = 0. \tag{9}$$

Equation (6) is hyperbolic if all three roots s_i are real and if there exist three linearly independent eigenvectors corresponding to s_i.

When $q_1' = q_2' = 0$, the characteristics (8) are

$$\left(\frac{dx}{dt}\right)_{1,2} = u \pm (gh)^{\frac{1}{2}}, \tag{10}$$

$$\left(\frac{dx}{dt}\right)_3 = 0, \tag{11}$$

which are written in terms of the physical variables.

Note that we have taken as characteristic parameters U, H the true velocity and depth u, h of the undisturbed steady and uniform flow, so that $u' = h' = 1$ and $Q_b = q_b$. Hence the numbers $\mathbf{F}$, $\mathbf{T}$, $\mathbf{J}$, $\mathbf{M}$, $\mathbf{H}$ turn out to be respectively equal to u^2/gh, L/uT, $c_b L u^2/gh^2$, q_b/uh and $\tau_0/|\tau|$.

The characteristics (10) give the velocity of propagation of a surface perturbation of the flow over a rigid bed. The third one appears because we have increased by one the order of the differential system.

If we suppose that $\mathbf{M} \ll 1$, all three roots s_i turn out to be real. They may most easily be determined by expanding in powers of $\mathbf{M}$:

$$s_i = s_i^{(0)} + \mathbf{M}s_i^{(1)} + \mathbf{M}^2 s_i^{(2)} + \ldots,$$

where the subscript i is used to number the roots and the superscript indicates the order of the approximation. The approximation of zeroth order is given by (10), (11). Substituting this series into (9) and considering only the first-order approximation we can write the characteristics as follows:

$$\left(\frac{dx}{dt}\right)_1 = v_1 = u + (gh)^{\frac{1}{2}} + \frac{q_1'\mathbf{F} + q_2'\mathbf{F}^2}{2\mathbf{F}^2 + 2\mathbf{F}^3}\mathbf{M}u + O(\mathbf{M}^2), \tag{12}$$

$$\left(\frac{dx}{dt}\right)_2 = v_2 = u - (gh)^{\frac{1}{2}} - \frac{q_1'\mathbf{F} - q_2'\mathbf{F}^2}{2\mathbf{F}^2 - 2\mathbf{F}^3}\mathbf{M}u + O(\mathbf{M}^2), \tag{13}$$

$$\left(\frac{dx}{dt}\right)_3 = v_b = \frac{q_1' - q_2'}{1 - \mathbf{F}^2}\mathbf{M}u + O(\mathbf{M}^2). \tag{14}$$

The first two characteristics give the velocity of propagation of a disturbance of the flow variables u, h (gravity wave) in an erodible-bed channel that can propagate downstream or upstream. As the additional terms due to the erodibility of the bed are of $O(\mathbf{M})$, gravity waves propagate with a velocity that is practically unaffected by the erodible bed. This can be taken as a justification of the well known computational techniques used to evaluate floods and progressive waves in open channels and natural streams with the bottom considered as rigid.

The characteristic (14) gives the velocity of propagation of a bed disturbance, and it is much smaller than the velocity v_1 of the downstream gravity wave. In general, v_1 is of the order of 1 m/sec, whereas v_b is of the order of 1 m/hour, i.e. both waves are practically decoupled. On the other hand, the wave velocity v_2 may yield surface waves moving upstream or downstream that are coupled with bed waves, as may be observed from (13). This particular aspect will be discussed in §4.1.

In the vicinity of critical flow ($\mathbf{F} = 1$), the wave velocities (13) and (14) increase rapidly without bounds. As the roots of (9) depend continuously on $\mathbf{F}$, it may be expected that real and bounded roots should exist when $\mathbf{F}^2 \to 1$. Hence the approximations (12)–(14) are no longer valid near critical flow, and their singularities can be attributed to the perturbation method. In the neighbourhood of critical flow, these characteristics are

$$v_1 = 2u + \tfrac{1}{4}(q_1' + q_2')\,\mathbf{M}u + O(\mathbf{M}^2), \tag{15}$$

$$v_2 = -v_b = -[\tfrac{1}{2}(q_1' - q_2')\,\mathbf{M}]^{\frac{1}{2}}\,u + O(\mathbf{M}), \tag{16}$$

as obtained again by the perturbation method, using

$$s_i = s_i^{(0)} + \mathbf{M}^{\frac{1}{2}} s_i^{(1)} + \mathbf{M}s_i^{(2)} + \ldots$$

as the series expansion for s_i.

It is not difficult to show that if the roots of (9) are all real, then

$$u-[gh(1+q_1'\mathbf{M})]^{\frac{1}{2}} < \left(\frac{dx}{dt}\right)_i < u+[gh(1+q_1'\mathbf{M})]^{\frac{1}{2}}$$

gives an upper and lower bound for these roots. The approximate roots (12) to (16) satisfy these inequalities.

When we choose for the bed discharge q_b the formula (4), the bed wave velocity is given by

$$v_b = \frac{(2+n)\,r}{1-\mathbf{F}^2}\mathbf{M}^*u+O(\mathbf{M}^2). \tag{17}$$

2.4. *Quasi-steady approximation* $\mathbf{M} \ll 1, \mathbf{T} \ll 1$

There is a possibility of transforming the shallow water equations (6) in such a way that gravity waves may be 'filtered' from them, so that only waves due to a bed disturbance appear. To discuss this possibility, let us consider a flow which has a characteristic time $T \gg 1$, i.e. the period of the velocity field is very large. Therefore $\mathbf{T} \ll 1$ and we can disregard the partial derivatives with respect to t' in the first two equations of system (6). Thus, the dimensionless velocity $u' = q'/h'$ turns out to be a function of h', where $q' = q/UH$ is the dimensionless flow discharge. As $\mathbf{M} \ll 1$, all terms of the last equation of (6) must be taken into account and the system (6) reduces to

$$\left(1-\frac{q'^2}{h'^3}\mathbf{F}^2\right)\frac{\partial h'}{\partial x'}+\frac{\partial e'}{\partial x'}+\mathbf{J}\frac{q'^2}{h'^{(3+n)}} = 0, \tag{18}$$

$$\left(-\frac{q'}{h'^2}q_1'+q_2'\right)\frac{\partial h'}{\partial x'}+\mathbf{Z}\frac{\partial e'}{\partial t'} = 0, \tag{19}$$

with dependent variables h' and e'. The number $\mathbf{Z} = \mathbf{T}/\mathbf{M} = LH/Q_bT$ expresses the ratio between the characteristic volume of the fluid that produces the erosion and the total volume of eroded material during the characteristic time T. $\mathbf{Z}$ is of order unity or larger when T is sufficiently large.

This is the quasi-steady approximation† of the shallow-water theory, which is valid only when the velocity u' and the depth h' vary so slowly in time that the fluid motion may be considered steady and $\mathbf{M} \ll 1$. Unsteadiness is due only to the motion of the particles of the bed. Similar equations, written in dimensional form, have been previously used by Exner (1925), Reynolds (1965) and Gradowczyk & Folguera (1965). Exner and Reynolds supposed, however, that q_b is a function only of the mean velocity u, i.e. $q_2' \equiv 0$.

The characteristics of the system (18), (19) are given by

$$\begin{vmatrix} 1-\dfrac{q'^2}{h'^3}\mathbf{F}^2 & 0 & 1 & 0 \\ -\dfrac{q'}{h'^2}q_1'+q_2' & 0 & 0 & \mathbf{Z} \\ dx' & dt' & 0 & 0 \\ 0 & 0 & dx' & dt' \end{vmatrix} = 0,$$

† Sometimes called 'hydraulic model'.

i.e. $$\left(\frac{dx}{dt}\right)_1 = v_b = \frac{q_1' - q_2'}{1-\mathbf{F}^2}\mathbf{M}u, \tag{20}$$

$$(dt)_2 = 0, \quad \text{i.e.} \quad t = \text{constant}, \tag{21}$$

since we have taken $u' = h' = 1$ as in §2.3.

It can be observed that gravity waves have now disappeared; (20) turns out to be equal, up to terms of $O(\mathbf{M}^2)$, to the characteristic (14) of the complete system (6). Therefore the quasi-steady equations can be considered as the 'filtered equations' from the shallow-water theory. Bed disturbances propagate now with the first-order approximation of the bed-velocity (14) of the complete theory and we have eliminated the awkward gravity waves. Note that surface waves, moving with the bed wave velocity, may propagate in this model, as will be shown in §§3 and 4.2.

It may be expected that quasi-steady problems, e.g. the scouring around an obstacle, could be described by (18), (19). Gradowczyk & Folguera (1965) integrated these equations using a numerical procedure. The computations are in fairly good agreement with experimental data, which gives support to the practical value of this theory. Our analysis justifies the quasi-steady approach in a more systematic manner, showing at the same time its limitations.

3. Linear stability

Let us consider the influence of the erodibility of the bed on the stability of one-dimensional open-channel flows. It is well known that open-channel flows may become unstable when the Froude number is sufficiently high; in this case the steady flow changes into a discontinuous periodic wave pattern of roll waves. This phenomenon, observed by many researchers, e.g. Cornish (1934), was first discussed theoretically by Jeffreys (1925). Later on, Thomas (1940) and Dressler (1949) discussed this problem further. All these authors considered the bed as rigid and flat. From these studies a critical Froude number $\mathbf{F}_c$ has been obtained which gives the limit of stable flows. When $\mathbf{F} > \mathbf{F}_c$, the flow must become unstable and roll waves are to be expected.

We shall investigate whether periodic waves of the form $\exp(ik)(x-vt)$ may propagate in free-surface flows over erodible beds using a linear stability analysis similar to that applied by Jeffreys.

Let us consider a steady flow $u' = h' = e' = \text{const.}$ which is perturbed in such a way that the new flow variables are $u' + \delta u'$, $h' + \delta h'$, $e' + \delta e'$, where $\delta u'(x', t') \ldots$ are the dimensionless perturbations introduced to the main flow as shown in figure 2. As the perturbed motion must fulfil system (6), we substitute $u' + \delta u' \ldots$ in it after neglecting terms $O(\delta u'^2)$ and higher.

If the original equations of the undisturbed flow are subtracted from the equations of the perturbed flow, a homogeneous linear system of partial differential equations in terms of the disturbances $\delta u'$, $\delta h'$, $\delta e'$ is finally obtained, which may be written in the matrix form

$$\mathbf{K}\,\delta\mathbf{v}' = 0, \tag{22}$$

where
$$\mathbf{K} = \begin{bmatrix} \mathbf{F}^2\left[\mathbf{T}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}\right]+2\mathbf{J} & \frac{\partial}{\partial x'}-(1+n)\,\mathbf{J} & \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial x'} & \mathbf{T}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'} & 0 \\ 2r\mathbf{M}^*\frac{\partial}{\partial x'} & -nr\mathbf{M}^*\frac{\partial}{\partial x'} & \mathbf{T}\frac{\partial}{\partial t'} \end{bmatrix},$$

$$\delta\mathbf{v}' = [\delta u', \delta h', \delta e'].$$

We take as characteristic parameters U, H the true values u, h of the steady and uniform flow ($u' = h' = 1$).

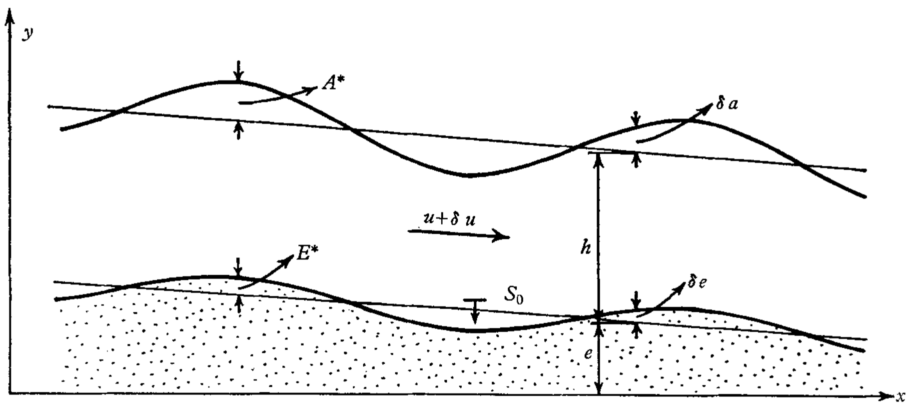

FIGURE 2. Symbols in stability analysis.

Since (22) is a linear system, it can be reduced to the single equation

$$(2+n)\,r\mathbf{M}^*\frac{\partial\phi^3}{\partial x'^3}-\mathbf{F}^2\mathbf{T}^3\frac{\partial\phi^3}{\partial t'^3}+(1+2r\mathbf{M}^*-\mathbf{F}^2)\,\mathbf{T}\frac{\partial^3\phi}{\partial t'\,\partial x'^2}$$
$$-2\mathbf{F}^2\mathbf{T}^2\frac{\partial^3\phi}{\partial t'^2\,\partial x'}+2\mathbf{J}\mathbf{T}^2\frac{\partial^2\phi}{\partial t'^2}+(3+n)\,\mathbf{J}\mathbf{T}\frac{\partial^2\phi}{\partial t'\,\partial x'} = 0, \tag{23}$$

where ϕ stands for $\delta u'$, $\delta h'$ or $\delta e'$. In order to obtain the condition for stability, consider first a neutral disturbance of the form

$$\phi = \phi^*\exp\{ik(x-vt)\}, \tag{24}$$

where $k = 2\pi/l$ is the wave-number of the perturbation, l is its wavelength, v its wave velocity and ϕ^* is the amplitude; both k and v are real. It is convenient to adopt k^{-1} as the characteristic length, so that

$$\mathbf{T} = 1/kuT, \quad \mathbf{J} = S_0/kh. \tag{25}$$

When (24) is substituted in (23), the imaginary part gives

$$v = \tfrac{1}{2}(3+n)\,u, \tag{26}$$

and the real part (with $\mathbf{F} = \mathbf{F}_c$) for this critical disturbance) gives

$$\mathbf{F}_c^2 = \frac{4}{(1+n)^2}+\frac{8}{(1+n)^2(3+n)}\,r\mathbf{M}^*. \tag{27}$$

It is observed from (26) that the wave velocity of the perturbation (24) is independent of the bed material and is equal to the velocity of a flood wave moving over a rigid and flat bottom. When $\mathbf{M}^* = 0$, the expression (27) reduces to the form given by Jeffreys (1925) or Dressler & Pohle (1953) for a rigid-bed channel.

It follows from (27) that the erodibility of the bed helps to stabilize the flow because the term due to the bed is positive. Since in this region $\mathbf{H} \ll 1$, the formula (27) can be written as

$$\mathbf{F}_c^2 \approx \frac{4}{(1+n)^2[1-C(d_m/h)^{3n/2}]}$$

when the Meyer-Peter formula $q_b \approx \chi|\tau|^{\frac{3}{2}}$ is used. C is a positive constant. Thus, if the relative roughness d_m/h is increased, so is $\mathbf{F}_c$. This is in agreement with experience; Forchheimer (1930) and Rouse (1938) explained that roll waves may be avoided by making the channel sufficiently rough.

If we compare this result with experiments reported by Guy *et al.* (1966), it follows that our analysis is in agreement with the experimental data when the contact-bed discharge q_b is larger than the suspended-material discharge q_s, i.e. the transported material is mostly carried near the bed. This occurs when the sand bed is sufficiently coarse, say $d_m \geqslant 0{\cdot}45$ mm. On the other hand, when $q_s \gg q_b$, i.e. very fine bed material, the suspended material seems to have a distabilizing effect on the flow since the observed values of $\mathbf{F}_c$ are even smaller than the one corresponding to the rigid bed condition.

This conclusion might be expected since the formula (4) used in the analysis represents only the contact-bed discharge. Our model is not sufficiently refined to describe the more complex situation $q_s \gg q_b$ because it would be necessary to know the distribution of velocity and the concentration of sediment along the y-axis in order to compute q_s, and this is not determined by the simple model described here.

To investigate the relationship that exists between the different wave amplitudes, it is assumed, as a special case of (24), that the neutral wave is of the form

$$\delta v' = [U^{*\prime}, H^{*\prime}, E^{*\prime}] \sin k(x-vt), \tag{28}$$

where $U^{*\prime} = U^*/u$, $H^{*\prime} = H^*/h$, $E^{*\prime} = E^*/h$ are the dimensionless real amplitudes. Substituting (28) into the second and third rows of (22), we obtain

$$U^* = (v'-1)\frac{u}{h}(A^*-E^*), \tag{29}$$

$$\left[1+\left(2-\frac{2+n}{v'}\right)r\mathbf{M}^*\right]E^* = \left(2-\frac{2+n}{v'}\right)r\mathbf{M}^*A^*, \tag{30}$$

where $v' = v/u$ and $A^* = H^*+E^*$ is the water surface amplitude shown in figure 2. These expressions are valid for $\mathbf{F} = \mathbf{F}_c$.

It follows, after substituting (26) into (30), that E^* is proportional to $\mathbf{M}^*A^*$. Since $\mathbf{M}^* \ll 1$ in most practical cases, it can be concluded that the bed is practically unaffected by the wave (28).

It may be of interest for further study to apply this stability analysis to the quasi-steady approximation discussed in §2.4. The corresponding equation for the function ϕ can be derived from (23) if we substitute in it $\mathbf{M}^* = \mathbf{T}/\mathbf{Z}^*$ where $\mathbf{Z}^* = (1-\mathbf{H})\,\mathbf{T}/\mathbf{M}$, then divide (23) by $\mathbf{T}$ and finally set $\mathbf{T} = 0$. This yields

$$(2+n)\,r\frac{\partial^3\phi}{\partial x'^3}+(1-\mathbf{F}^2)\,\mathbf{Z}^*\frac{\partial^3\phi}{\partial t'\,\partial x'^2}+(3+n)\,\mathbf{J}\mathbf{Z}^*\frac{\partial^2\phi}{\partial t'\,\partial x'} = 0. \tag{31}$$

A non-vanishing neutral perturbation of the form (24) or (28) can only propagate if $\mathbf{J} = 0$ and $\mathbf{F} \neq 1$. The velocity of this perturbation turns out to be equal to the characteristic (20) of the quasi-steady equations. Reynolds (1965) obtained a similar result by means of a somewhat different approach.

The relations between the quasi-steady wave amplitudes are given by

$$U^* = -\frac{u}{h\mathbf{F}^2}A^*, \quad E^* = (1-\mathbf{F}^{-2})\,A^*, \qquad (32), (33)$$

as can be readily verified.

In this way we have obtained a neutral bed and surface wave moving with the same bed wave velocity v_b. The significance of (32) and (33) will be discussed in §4.2.

To determine what happens when $\mathbf{F} \gtrless \mathbf{F}_c$, it is necessary to consider not only disturbances periodic in time and space but all possible perturbations. As we are interested in disturbances that are periodic in space, let us assume that

$$\phi = \phi^* \exp\,(at'/\mathbf{T}-ikx'), \tag{34}$$

where a is complex and k is real. A perturbation is stable if $\mathrm{Re}\,(a) < 0$.

The substitution of (34) into (23) yields the cubic

$$a^3+(p_1+ir_1)\,a^2+(p_2+ir_2)\,a+(p_3+ir_3) = 0, \tag{35}$$

where
$$p_1 = \frac{2\mathbf{J}}{\mathbf{F}^2}, \quad p_2 = (1+2r\mathbf{M}^*-\mathbf{F}^2)\frac{k^2}{\mathbf{F}^2}, \quad p_3 = 0,$$
$$r_1 = -2k, \quad r_2 = -(3+n)\frac{k\mathbf{J}}{\mathbf{F}^2},$$
$$r_3 = -(2+n)\,rk^3\frac{\mathbf{M}^*}{\mathbf{F}^2}.$$

The roots of (35) are a_1, a_2, a_3. Then $\mathrm{Re}\,(a_i) < 0$ ($i = 1, 2, 3$) if and only if

$$p_1 > 0,$$
$$\begin{vmatrix} p_1 & p_3 & -r_2 \\ 1 & p_2 & -r_1 \\ 0 & r_2 & p_1 \end{vmatrix} > 0,$$

and
$$\begin{vmatrix} p_1 & p_3 & 0 & -r_2 & 0 \\ 1 & p_2 & 0 & -r_1 & -r_3 \\ 0 & p_1 & p_3 & 0 & -r_2 \\ 0 & r_2 & 0 & p_1 & p_3 \\ 0 & r_1 & r_3 & 1 & p_2 \end{vmatrix} > 0,$$

(Frank 1946). The first condition will be always satisfied. The remaining conditions on simplification imply that the disturbances will be stable (unstable) when $\mathbf{F} < \mathbf{F}_c$ ($\mathbf{F} > \mathbf{F}_c$).

4. Coupled one-dimensional bed and surface waves

We have shown in §§2 and 3 how gravity waves may be affected by the erodibility of the bed; these waves do not represent any of the typical coupled bed and surface waves that appear in streams and flumes as described in §1. An appropriate dimensional analysis will show how the linear stability analysis given in §3 can yield bed configurations similar to those reported by experimentalists and discussed analytically by Kennedy and Reynolds.

4.1. *Neutral waves*

Let us write the ratio $\mathbf{T}/\mathbf{J}$ between numbers (25) as h/LS_0, where $L = uT$ is the distance travelled downstream by a particle of the flow moving with the velocity u after a time T. If T is sufficiently large for the channel depth h to be much smaller than the depth fall of the particle LS_0, then $\mathbf{T}/\mathbf{J} \ll 1$. Besides, if k^{-1} is of order of L it follows that $\mathbf{J} \gg 1$, and $\mathbf{T}$ is of order unity. Consequently, (23) can be reduced to a first order equation

$$\frac{\partial \psi}{\partial t}+\frac{3+n}{2}u\frac{\partial \psi}{\partial x} = 0, \tag{36}$$

which is written in terms of the true variables and with $\psi = \partial\phi/\partial t$. Therefore the wave (36) travelling with velocity (26) can be considered as a 'kinematic surface wave' for a time $T \gg h/uS_0$, if its wavelength l is sufficiently large.† Thus, the asymptotic behaviour of this wave is independent of the bed material.

On the other hand, if we consider bed and surface waves whose wavelengths are so small that h is much larger than the depth fall S_0/k, it follows that $\mathbf{J} \ll 1$. Consequently, it will be possible to neglect in (23) those terms which are multiplied by $\mathbf{J}$.

This assumption will allow us to discuss coupled bed and surface waves by the method of §3. It is easy to show, after substituting (24) into (23) and setting $\mathbf{J} = 0$, that the only neutral waves that can propagate in a one-dimensional flow turn out to be those waves which move with the velocities (12)–(16) and fulfil the inequality $kh \gg S_0$, i.e. relatively short waves. Longer waves should be practically decoupled from the bed.

As we are interested in slow coupled waves, let us consider first neutral waves of the form (28) moving with velocity (14) or (16). Explicit relations between amplitudes E^*, U^* and A^* are immediately obtained after substituting $v' = v_b/u$ into (29), (30). It follows that for flows with $\mathbf{F} \neq 1$

$$E^* = \left[1-\frac{1}{\mathbf{F}^2}+\frac{2r}{\mathbf{F}^4}\mathbf{M}^*+O(\mathbf{M}^2/\mathbf{F}^4)\right]A^*, \tag{37}$$

$$U^* = -\frac{u}{h}\left[\frac{1}{\mathbf{F}^2}-\frac{2r\mathbf{M}^*}{\mathbf{F}^4}-\frac{2+n}{1-\mathbf{F}^2}\frac{r\mathbf{M}^*}{\mathbf{F}^2}+O(\mathbf{M}^2/\mathbf{F}^2)\right]A^*, \tag{38}$$

† Kinematic waves moving over rigid beds have been thoroughly discussed by Lighthill & Whitman (1955). 'Kinematic bed waves' have been considered by Gradowczyk, Maggiolo & Raggi (1967).

and that for critical flows ($\mathbf{F} \approx 1$)

$$E^* = -\{[2(2+n)\,r\mathbf{M}^*]^{\frac{1}{2}} + 2(2+n)\,r\mathbf{M}^* + O(\mathbf{M}^{\frac{3}{2}})\}\,A^*, \tag{39}$$

$$U^* = -\frac{u}{h}\left\{1 + \left[\left(1+\frac{n}{2}\right) r\mathbf{M}^*\right]^{\frac{1}{2}} - (4+n)\,r\mathbf{M}^* + O(\mathbf{M}^{\frac{3}{2}})\right\} A^*. \tag{40}$$

Three different wave configurations, travelling with the velocity v_b, are obtained from the above relations.

(i) Subcritical flows ($\mathbf{F} < \mathbf{F}_f$). The bed and surface waves move downstream and are out of phase; the flow accelerates over the crests of the bed wave and decelerates over their troughs so that the total shear stresses $\tau + \delta\tau$ transmitted by the flow to the bed are higher (lower) where the fluid accelerates (decelerates). This produces the downstream advance of the bed-wave-pattern because the bed particles are eroded from the crests and deposited on the troughs. This type of wave behaviour resembles the ripples or dunes of the lower régime (α). When the velocity of the flow is increased and all the flow parameters are kept constant, the bed amplitude E^* decreases to $E^* = O[(\mathbf{M}/\mathbf{F})^2]\,A^*$ when $\mathbf{F}$ reaches the critical value $\mathbf{F}_f^2 = 1 - 2r\mathbf{M}^*$ and the bed configuration changes from a sinusoidal wave pattern into a practically flat bed. Thus, $\mathbf{F}_f$ furnishes the upper limit to the subcritical flow.

(ii) Supercritical flows ($\mathbf{F} > 1$). Both bed and surface waves are in phase and move upstream. The mechanism of subcritical waves (i) is reversed; the flow accelerates over the troughs and decelerates over their crests. The upstream advance of this bed wave can also be explained because the bed particles are now eroded from the troughs where the shear stresses are the highest and deposited in the crests. This wave is similar to those antidunes moving upstream which appear in the upper régime (γ).

(iii) Transition flow $\mathbf{F}_f \leqslant \mathbf{F} \leqslant 1$. When the critical flow condition $\mathbf{F} = 1$ is achieved, E^* is proportional to $\mathbf{M}^{*\frac{1}{2}}A^*$ as is evident from (39), and the bed wave amplitude turns out to be much smaller than the water-wave-amplitude since $\mathbf{M}^* \ll 1$. The bed undulations are very small and they are described as 'wash-out dunes'. Hence, this zone is the transition régime (β) which moulds beds ranging from the typical of the lower régime (i) to those of the flat bed zone of the upper régime (ii).

What happens if we consider relatively short waves $kh \gg S_0$ travelling with velocities v_1 or v_2? Since $v_1 \gg |v_b|$ in the domain of interest, the wave travelling with the velocity v_1 is a downstream surface wave practically decoupled from the bed. When a wave travelling with the velocity v_2 is considered, three situations of interest occur.

(iv) Subcritical antidunes: it is possible to find a Froude number $\mathbf{F}_1 < 1$ that yields a velocity $v_2 = O(v_b) < 0$. This corresponds to bed and surface waves which are in phase and move upstream, i.e. antidunes in subcritical flows.

(v) Downstream antidunes: similarly a Froude number $\mathbf{F}_2 > 1$ that yields a velocity $v_2 = O(v_b) > 0$ can also be found. This leads to antidunes moving downstream in supercritical flow, which are in phase with the surface wave if

$$0 < v_2 < (2+n)\,r\mathbf{M}^*u/(1+2r\mathbf{M}^*).$$

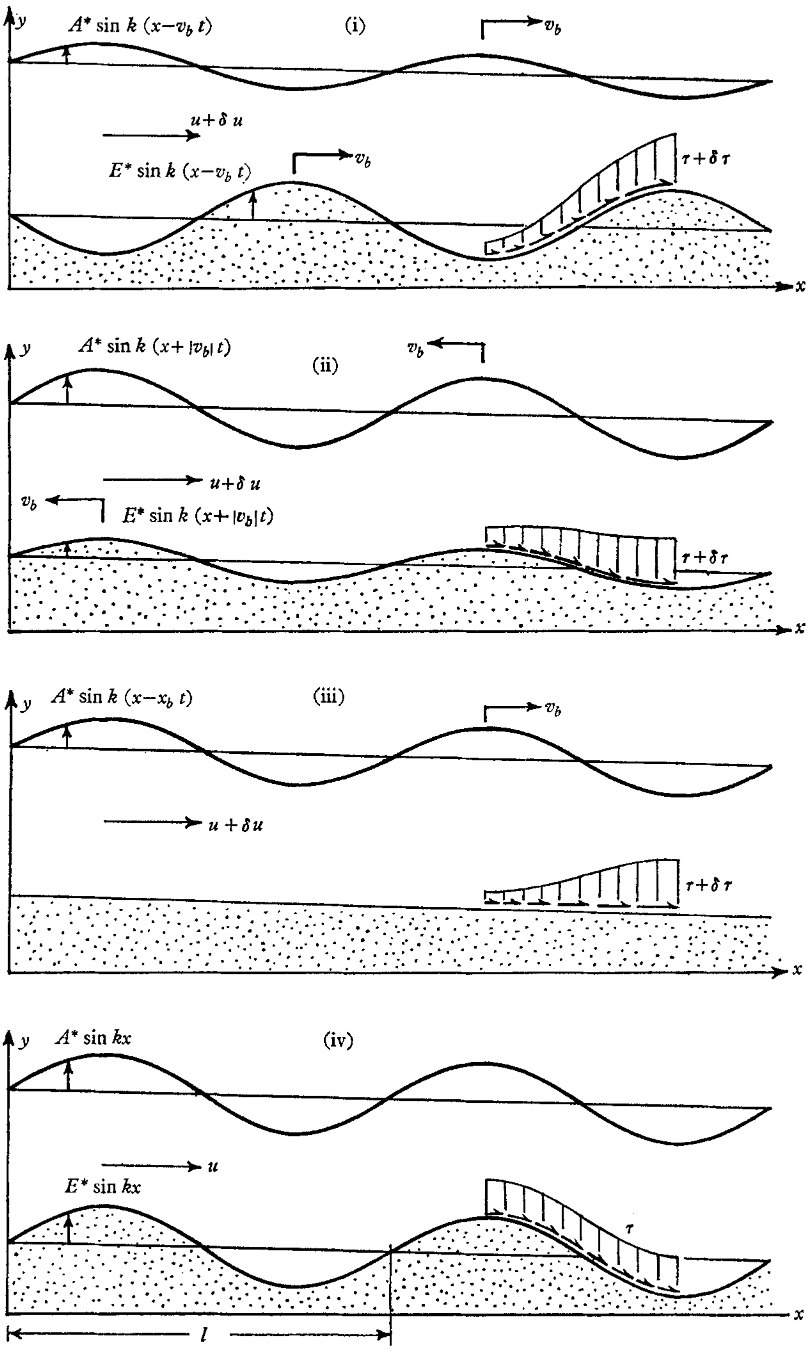

Figure 3. Different bed forms. (i) Sinusoidal wave pattern $\mathbf{F} < \mathbf{F}_f$. (ii) Antidunes moving downstream $1 < \mathbf{F} < \mathbf{F}_c$. (iii) Transition régime $\mathbf{F}_t \leqslant \mathbf{F} \leqslant 1$. (iv) Stationary antidunes $1 < \mathbf{F} = \mathbf{F}_3 < \mathbf{F}_c$.

(vi) Stationary waves: when $\mathbf{F} = \mathbf{F}_3 > 1$, it is possible to form a wave as a linear combination of the preceding waves (ii), (v) which moves with velocity

$$\Delta v = v_2 - v_b = O(\mathbf{M}^2).$$

This is practically a stationary wave, where $E^* \approx A^*$, $U^* \approx 0$. Stationary bed and surface waves are in phase; the total shear stresses $\tau + \delta\tau \approx \tau$ are constant along the bed. This explains why these waves remain in their original position, because no net difference of eroded and deposited bed material can be produced by a constant distribution of shear stresses along the bed.

All the above waves (i)–(vi) have been observed in experiments, e.g. Vanoni & Brooks (1957) and Guy, Simons & Richardson (1966). The bed configurations (i), (ii), (iii), (vi) are shown in figure 3.

Case	Flow régime	Froude number	Bed configuration	Movement of bed features
(i)	Lower	$0 < \mathbf{F} < \mathbf{F}_f$	Sinusoidal pattern	Downstream
(iii)	Transition	$\mathbf{F}_f \leqslant \mathbf{F} \leqslant 1$	Flat bed or small undulations	—
(ii)	Upper	$1 < \mathbf{F} < \mathbf{F}_c$	Antidunes	Upstream
(iv)	Upper	$\mathbf{F}_1 < \mathbf{F} < \mathbf{F}_c (\mathbf{F}_1 < 1)$	Antidunes	Upstream
(v)	Upper	$\mathbf{F}_2 < \mathbf{F} < \mathbf{F}_c (\mathbf{F}_2 > 1)$	Antidunes	Downstream
(vi)	Upper	$1 < (\mathbf{F} = \mathbf{F}_3) < \mathbf{F}_c$	Antidunes	Stationary
(vii)	Upper	$\mathbf{F}_c < \mathbf{F}$	Chutes and pools	—

TABLE 1. Summary of bed configurations, which are valid for relatively short waves $1 \gg (kh)^2$, $kh \gg S_0$

Finally, it is to be noticed that when $\mathbf{F} > \mathbf{F}_c$, the free-surface may become unstable as shown in §3, therefore discontinuous surface waves as breaking waves, bores and hydraulic jumps may appear. This situation has been observed in experimental work and natural streams and named 'chutes and pools' or violently breaking antidunes. The channel consists of a series of pools, in which the flow may be subcritical or supercritical, connected by steep chutes in which the flow is supercritical. The concentration of suspended material can be rather high for this configuration.

These highly coupled waves must be discussed by non-linear methods, therefore, the upper régime configurations (ii), (iv), (vi) are limited by the condition $\mathbf{F} < \mathbf{F}_c$. No limiting condition of this type has been yet explicitly given.

The various configurations derived from shallow-water theory are summarized in table 1.

In general, the analytical predictions are in agreement with waves observed in experimental flumes and natural streams, at least from a qualitative point of view, especially in the transition and upper régimes. However, this linear analysis does not take into account all aspects of the lower régime, where four bed configurations have been reported: ripples ($d_m < 0{\cdot}6$ mm), incipient flat beds ($d_m > 0{\cdot}6$ mm), ripples on dunes, dunes. Our analysis gives one type of bed form: a sinusoidal wave pattern.

If we apply the above linearized stability analysis to the quasi-steady approximation and set $\mathbf{J} = 0$ in (23), only the configurations (i) and (ii) are again obtained for $\mathbf{F} \neq 1$, as can be observed from (32), (33). It can be concluded that the quasi-steady shallow-water approximation is not an appropriate model for the study of coupled bed and surface neutral waves.

The above results as already stated have been obtained on the assumption $\mathbf{H} < 1$. If $\mathbf{H} \geqslant 1$, the bed remains at rest. Since temperature variations modify the viscosity of the stream, this will change the value of τ_0^*. This may help to explain why some rivers, at similar discharges, have dunes in summer when the stream fluid is warm and less viscous and a flat bed in winter.

4.2. *Comparison between potential and shallow-water theory*

Let us relate our results to those obtained by Kennedy (1963). It is to be noted that his principal variables U, d, k, U_b, $\mathbf{F}$, $\xi(x,t)$, $A(t)$, $\eta(x,t)$, $a(t)$, G/B, n, x, t are respectively equal to our variables u, h, k, v_b, $\mathbf{F}$, δa, $A^*(t)$, δe, $E^*(t)$, q_b, r, x, t. Kennedy's formulae are expressed in the notation of this paper and the formula numbers followed by the letter K corresponds to his work.

Potential flow analysis is based on the parameter D, which is a characteristic water depth, and is defined implicitly by†

$$kh\mathbf{F}^2 = \tanh kD. \tag{9K}$$

When $D - h \gtrless 0$ the bed configuration corresponds respectively to antidunes (waves moving upstream) or dunes (waves moving downstream), as it can be observed from the bed wave velocity for neutral waves ($\delta = 0$)

$$v_b = -2r\mathbf{M}^* ukh \coth kh \left(\frac{D}{h} - 1\right) \equiv 2r \frac{1 - \mathbf{F}^2 kh \tanh kh}{\dfrac{\tanh kh}{kh} - \mathbf{F}^2} \mathbf{M}^* u \tag{21K}$$

and the relation between the bed and surface neutral amplitudes

$$E^* = [1 - (kh\mathbf{F}^2)^{-1} \tanh kh] \cosh khA^*. \tag{16K}$$

We will compare now the shallow-water and quasi-steady potential flow theories. If we assume that $kh\mathbf{F}^2 < 1$, equation (9K) can be expanded into series

$$D = h\mathbf{F}^2\{1 + O[(kh\mathbf{F}^2)^2]\}. \tag{41}$$

Similarly, since $\mathbf{F}$ is of order unity or smaller, formulae (21K) and (16K) can be developed into series

$$v_b = 2\left\{\frac{1}{1 - \mathbf{F}^2} + O[(kh)^2]\right\} r\mathbf{M}^* u, \tag{42}$$

$$E^* = \{(1 - \mathbf{F}^{-2}) + O[(kh)^2]\} A^*. \tag{43}$$

† Reynolds (1965) has shown that it is not necessary to introduce D in the analysis; his expressions for (16K) and (21K) are, however, similar to those derived by Kennedy.

Hence the two quasi-steady theories yield the same neutral waves when $\delta = 0$ and terms $O[(kh)^2]$ and higher are disregarded.† In addition, the ratio of $D/h \gtrless 1$ is similar to the inequality $\mathbf{F}^2 \gtrless 1$. When $D \to h(\mathbf{F}^2 \to 1)$, $v_b \to \infty$, a result that seems to be a peculiarity of the two quasi-steady approximations.

The singularities of the two quasi-steady theories are removed by the complete shallow-water theory because v_b has the finite value (16) in critical flow.

This comparison allows us to discuss the range of validity of the shallow-water theory and the potential theory. A lower bound given in § 4.1 indicates that coupled bed and surface waves may appear when $kh \gg S_0$; an upper bound which limits the applicability of shallow-water theory is $1 \gg (kh)^2$. Therefore infinitesimal coupled bed and surface waves may be described by shallow-water theory only if the inequalities

$$1 \gg (kh)^2, \quad kh \gg S_0 \tag{44}$$

are satisfied, as indicated in table 1.

The region bounded by (44) covers all the régimes (α), (β), (γ) mentioned in §§ 1 and 4.1 with the exception of ripples, which are characterized by $kh > 1$. The potential theory may be used to describe coupled waves in the zone

$$\mathbf{F}^{-2} > kh, \quad kh \gg S_0. \tag{45}$$

The lower bound must be explicitly stated, because potential theory does not allow for long surface waves like kinematic surface waves; the upper bound is given by the formula (30K). According to Reynolds's formula (19R), this upper bound should be $\coth(kh)/\mathbf{F}^2 kh$.

A numerical comparison between the reduced bed-wave velocities and bed amplitudes of the two theories is shown in figure 4. The agreement is good when $(kh)^2 \ll 1$.

4.3. *The growth of bed and surface waves*

Let us discuss the growth of a bed perturbation generated at $t = 0$. The equation (23) may be appropriate for this study, which requires the integration of an initial-value problem with $\phi(x', 0)$ given. Instead of following this general method, we assume that the perturbation vector (28) conserves its initial form but amplitudes $U^*(t)$, $H^*(t)$, $E^*(t)$ are now functions of t to be determined. As we shall relate our results to Kennedy's findings, we suppose that $\mathbf{T} \ll 1$ and $\mathbf{J} \ll 1$; the former is equivalent to the assumption used by Kennedy (1963, p. 526), who supposed that $H^*(t)$ is a slowly varying function of t; the latter is a basic hypothesis of potential flow theory. In addition, the phase shift δ, which is the distance by which the local bed discharge lags behind the local velocity at the bed, will be introduced.‡ Although no conclusive experimental result has shown the existence of such a lag, we use this mechanism as a free parameter that may take into account those aspects of real motion which are not considered in the mathematical model.

† The missing factor n in (42) is due to the fact that Kennedy assumed that $q_b = q_b(u)$.

‡ Note that the assumption $\mathbf{M} \ll 1$ is implicitly used in this section so that the analysis corresponds to the quasi-steady approximation of § 2.4.

With these considerations, it follows that the amplitude δe must satisfy the equation

$$\left[\frac{\partial \delta e}{\partial t}\right]_x + v_b \left[\frac{\partial \delta e}{\partial x}\right]_{x-\delta} = 0, \tag{46}$$

which is written in terms of the physical variables. The symbolism $[\;]_x$ means that the term between brackets is to be evaluated at x.

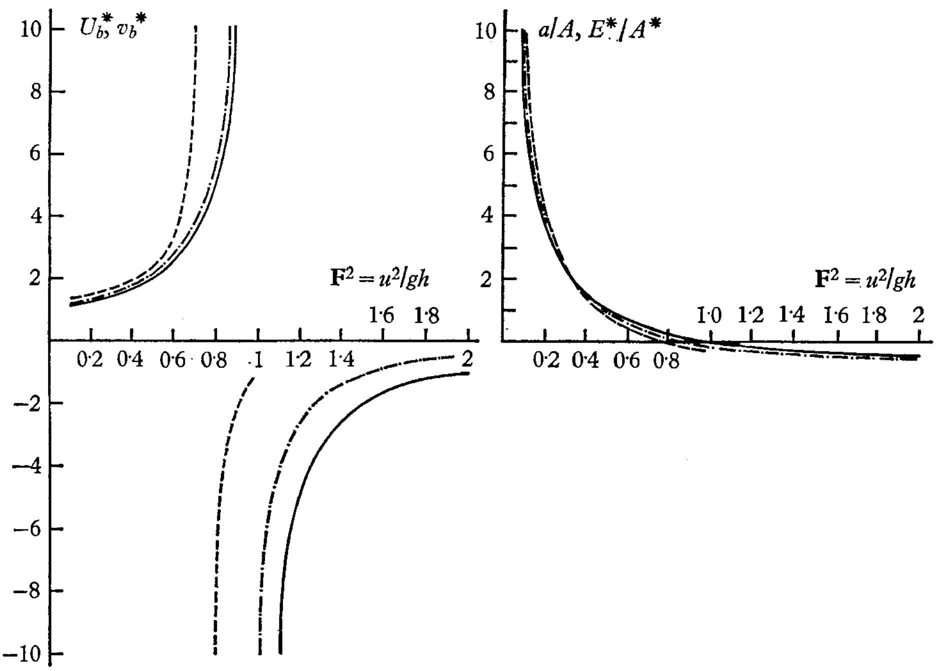

FIGURE 4. Comparison between the reduced wave velocities $v_b^* = v_b/(2+n)\mathbf{M}^*u$ and the bed wave amplitude ratio E^*/A^* of the quasi-steady potential and shallow-water theories, for different values of kh. The potential solutions are interrupted when the growth condition $\mathbf{F}^2 = 1/kh$ is achieved. The line that corresponds to $kh = 0{\cdot}1$ is not drawn because it is practically coincident with the shallow water approximation. ———, quasi-steady shallow-water approximation; –·–·–, $kh = 0{\cdot}5$; – – –, $kh = 1$ (quasi-steady potential theory).

Setting $\delta e = E^*(t) \sin k(x - vt)$ in (46), where v is the wave velocity, gives an ordinary differential equation for $E^*(t)$, the solution of which yields

$$v = v_b \cos k\delta, \tag{47}$$

$$E^*(t) = E^*(0) \exp\left[-ktv_b \sin k\delta\right]. \tag{48}$$

Equations (47) and (48) are similar to Kennedy's equations. If terms $O[(kh)^2]$ and smaller are disregarded, the factor $\mathbf{F}^2 - 1$ can be taken instead of $D - h$ as the difference that determines whether bed and surface beds are in phase or not. Consequently, the five bed forms shown by Kennedy (1963, p. 529) in table 1 can be obtained with this approach. These five bed forms are as follows: sinusoidal pattern moving downstream, flat beds and antidunes moving upstream, moving downstream and stationary.

These five configurations have already been discovered in §4.1 with the help of the shallow-water theory and without using δ. This suggests that δ may be taken as a free-parameter that modifies some aspects of the quasi-steady approximation in order to approach the complete theory.

This interpretation leads to the following question: is it possible to obtain the same wave configurations by means of a potential flow theory which does not assume quasi-steadiness and without the lag δ? The answer to this question may help to clear up the meaning of the parameter.†

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† Calculations performed after this manuscript was sent to the press showed that the waves of table 1 can be obtained by means of the fully unsteady potential theory without the use of δ.

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